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Topological quantum phase transition from a fermionic integer quantum Hall phase to a bosonic fractional quantum Hall phase through a *p*-wave Feshbach resonance

Shiuan-Fan Liou,[1] Zi-Xiang Hu,[2] and Kun Yang[1]
[1]*National High Magnetic Field Laboratory, Florida State University, Tallahassee, Florida 32306, USA*
[2]*Department of Physics, Chongqing University, Chongqing 401331, People's Republic of China*
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We use exact diagonalization to study the quantum phases and phase transitions when a single species of fermionic atoms at a Landau level filling factor $\nu_f = 1$ in a rotating trap interact through a p-wave Feshbach resonance. We show that under a weak pairing interaction, the system undergoes a second-order quantum phase transition from a $\nu_f = 1$ fermionic integer quantum Hall (FIQH) state at positive detuning, to a $\nu_b = \frac{1}{4}$ bosonic fractional quantum Hall (BFQH) state at negative detuning. However, when the pairing interaction increases, a new phase between them emerges, corresponding to a fraction of fermionic atoms staying in a coherent superposition of a bosonic molecule state and an unbound pair. The phase transition from the FIQH phase to the new phase is of second order and that from the new phase to BFQH phase is of first order.

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Introduction. Topological phases of matter are of tremendous current interest. Equally important, but perhaps less studied thus far (especially on the experimental side), are quantum phase transitions (QPTs) between different topological phases. This is because tuning the system through such QPTs requires great control of certain knobs, which are not always available in electronic condensed matter systems. Recently, there has been intense activities in topological phases realized in trapped cold atom systems [1–5]. The advantages of these systems include the availability of qualitatively different types of interparticle interactions, and the great abilities experimentalists have to control them. This allows for detailed studies of QPTs in such systems. Among such activities are attempts to realize quantum Hall (QH) states [6,7], which were the very first topological phases discovered.

In electronic systems, QPTs between different QH phases are usually driven by changing magnetic field, and disorder plays a dominant role at these transitions. In fact, the field-driven transition can often be mapped to a transition at a fixed field but driven by disorder strength instead. The presence of disorder, especially when combined with an electron-electron interaction, makes such transitions extremely hard to study theoretically. Over the years theorists instead have studied possible QH transitions driven by interactions *without* disorder [8–10]; in most cases these require an additional periodic potential that is not available in electronic systems.

It was pointed out by Yang and co-workers [11,12] that in cold atom systems, one can realize some of the closely related QH transitions with *neither* disorder *nor* a periodic potential. Instead, what drives the transitions are Feshbach resonances (FRs), across which two fermions bind into a bosonic molecule. The two phases involved are a fermionic integer QH phase on the atomic side (positive detuning), and a bosonic fractional QH phase on the molecular side (negative detuning). One such phase transition [11] driven by s-wave Feshbach resonance has been confirmed numerically [13]. The other transition from a $\nu_f = 1$ fermionic integer quantum Hall (FIQH) phase to a $\nu_b = \frac{1}{4}$ bosonic fractional quantum Hall (BFQH) was proposed by Barlas and Yang [12], where they studied this phase transition using quantum field theory methods, and argued that it is a second-order QPT in the (2+1)-dimensional Ising universality class.

In this Rapid Communication we use the exact diagonalization (ED) method to study the model of Barlas and Yang [12]. The purpose of such a numerical study is threefold. First, while field theory approaches may determine the possible phases, they cannot determine the actual location of the phase boundary; the latter needs to be determined by calculations based on microscopic models. Second, while field theory can tell if a second-order phase transition is possible, in general it does not guarantee it must happen in a specific microscopic model; in fact, first-order transitions are *always* possible. Lastly, since field theory only keeps certain long-wavelength/low-energy degrees of freedom, it might miss certain phases. Indeed, through our numerical work, we (i) determine the phase diagram quantitatively, (ii) confirm that the direct transition from FIQH phase to BFQH phase is of second order, and (iii) perhaps most importantly, we find an intermediate phase between FIQH and BFQH phases in certain regions of the phase diagram.

To investigate this QPT, we consider a single species of fermions confined to two dimensions and under either a rotation or synthetic gauge field, whose effects mimic that of a strong magnetic field. We focus on the case where the Landau level filling factor $\nu_f = 1$. When the system is tuned through p-wave FR, two fermions can pair up and form a p-wave bosonic molecule with twice the "charge." We assume the Landau level spacing is large enough such that we can assume all particles are confined to their respective lowest Landau level. The system can thus be described by the following Hamiltonian on a disk (which is the geometry of our choice for a finite-size numerical study, where total angular momentum is a good quantum number due to rotational symmetry),

$$H = \delta \sum_m \left(b_m^\dagger b_m - 2 f_m^\dagger f_m \right) + \left(\sum_{m_1,m_2,m_3} g_{m_1,m_2,m_3} b_{m_1}^\dagger f_{m_2} f_{m_3} + \text{H.c.} \right)$$

$$+\sum_{m_1,m_2,m_3,m_4} v^{(0)}_{m_1,m_2,m_3,m_4}\, b^\dagger_{m_1} b^\dagger_{m_2} b_{m_3} b_{m_4}$$
$$+\sum_{m_1,m_2,m_3,m_4} v^{(2)}_{m_1,m_2,m_3,m_4}\, b^\dagger_{m_1} b^\dagger_{m_2} b_{m_3} b_{m_4}, \qquad (1)$$

where $\{m_i\}$ are angular momentum quantum numbers of the lowest Landau level single-particle orbitals, with b_m ($b^\dagger_m$) and f_m ($f^\dagger_m$) the corresponding annihilation (creation) operators for bosons and fermions. The first term above is the chemical potential term in which δ is the detuning corresponding to the energy difference between a pair of fermions and a boson; this term determines whether the atoms should pair up to form molecules or stay unbound energetically.

The second term depicts the pairing interaction through p-wave FR. The matrix element $g_{m_1,m_2,m_3} = g\,\delta_{m_1,M}\,\langle 1,M|m_2,m_3\rangle$, where g represents the strength of the pairing interaction in the p-wave channel (corresponding to width of the p-wave FR), $|m_2,m_3\rangle$ is a two-body state with the two particles having angular momentum m_2 and m_3, respectively, and $|1,M\rangle$ is a two-body state with their relative angular momentum equal to 1 and center-of-mass angular momentum M. This matrix element indicates that only those pairs of fermions whose relative angular momentum $\Delta m = 1$ can pair up, and the consequent composite boson has an orbital angular momentum $m_1 = M = m_2 + m_3 - 1$. This term conserves the angular momentum because the p-wave bosonic molecule also carries an internal momentum $+1$. The matrix element $\langle 1,M|m_2,m_3\rangle$ is a special case of

$$\begin{aligned}
&\langle \Delta m,M|m_1,m_2\rangle \\
&\equiv \frac{1}{\sqrt{(2\pi)^4\, 2^{(\Delta m+M+m_1+m_2)}\,\Delta m!\,M!\,m_1!\,m_2!\,l^8}} \\
&\quad\times \int d^2z_1 \int d^2z_2 \left(\frac{z_1^*-z_2^*}{\sqrt{2}l}\right)^{\Delta m}\left(\frac{z_1^*+z_2^*}{\sqrt{2}l}\right)^{M} \\
&\quad\times \left(\frac{z_1}{l}\right)^{m_1}\left(\frac{z_2}{l}\right)^{m_2} e^{-\frac{|z_1|^2+|z_2|^2}{2l^2}} \\
&= \sqrt{\frac{\Delta m!\,M!}{(2\pi)^4\,2^{\Delta m+M}\,m_1!\,m_2!}} \\
&\quad\times \sum_n (-1)^{\Delta m-n}\, C_n^{m_1}\, C_{\Delta m-n}^{m_2}\, \delta_{\Delta m+M,m_1+m_2}, \qquad (2)
\end{aligned}$$

where $z_a \equiv x_a + iy_a$ is the complex coordinate of the ath particle on a disk in the lowest Landau level $d^2z_a = dx_a dy_a$, l is the corresponding magnetic length, and Δm is the relative angular momentum of a pair. The sum of n is summing over all natural numbers bounded at $\max(0,\Delta m - m_2) \leqslant n \leqslant \min(\Delta m, m_1)$. Note that the bosonic magnetic length square l_b^2 is half of the fermionic magnetic length square l_f^2 ($l_b^2 = l_f^2/2$) in our model due to the doubled charge of the bosons.

The last two terms correspond to *repulsive* interactions between bosons. In order to stabilize the BFQH state with $\nu_b = \frac{1}{4}$, we consider the zeroth order and the second order of Haldane pseudopotentials [14]. The matrix element $v^{(\alpha)}_{m_1,m_2,m_3,m_4} = v^{(\alpha)} \sum_M \langle m_1,m_2|\alpha,M\rangle\langle\alpha,M|m_3,m_4\rangle$, where $v^{(\alpha)}$ is the strength of the αth order Haldane pseudopotential. In our calculation, we simply use $v^{(0)} = v^{(2)} = 1$.

In this model, the total charge ($N_{\rm tot}$) and the total angular momentum ($M_{\rm tot}$) are good quantum numbers. The total charge is the sum of the number of fermions and twice the number of bosons,

$$N_{\rm tot} = 2N_b + N_f = \sum_m (2b^\dagger_m b_m + f^\dagger_m f_m). \qquad (3)$$

The prefactor 2 in the bosonic part comes from the fact that each bosonic molecule has twice the charge of the fermion. And the total angular momentum is the sum over all angular momenta of orbitals occupied by bosons and fermions plus the number of bosons since each boson has one internal angular momentum,

$$M_{\rm tot} = \sum_m [(m+1)b^\dagger_m b_m + m f^\dagger_m f_m]. \qquad (4)$$

In our numerical calculation, we use these two quantum numbers to label the sector in which we perform our calculations.

There are two limits of this Hamiltonian in Eq. (1). For $\delta > 0$ and $|\delta| \gg g$, a boson costs more energy than an unbound pair of fermions, so the ground state will be dominated by fermions and the system forms a FIQH state at $\nu_f = 1$, which does not require interactions between fermions to stabilize (and that is the reason why we do not include a fermion interaction in our model). On the other hand, when $\delta < 0$ and $|\delta| \gg g$, it is energetically favorable for fermions to pair into bosonic molecules. The resultant boson filling factor is $\nu_b = \frac{1}{4}$, resulting in a Laughlin-type BFQH state with the boson-boson interaction we introduced. We can easily distinguish between these two phases by inspecting their low-energy spectra, as we now turn to.

In Fig. 1, we show the energy spectra for a system with $N_{\rm tot} = 10$ given ten fermionic orbitals and 20 bosonic orbitals with $M_{\rm tot}$ from 44 to 48. Under a FIQH limit, the system will have the lowest-energy state only when it forms a FIQH state in which there are no bosons and all fermionic orbitals are occupied, namely, $M_{\rm tot} = M_{\rm gs}$, with

$$M_{\rm gs} = \frac{N_{\rm tot}(N_{\rm tot}-1)}{2}. \qquad (5)$$

$M_{\rm gs}$ is 45 in this case for $N_{\rm tot} = 10$. Because we only give the least number of fermionic orbitals, which is ten here for the FIQH state, no edge states should show up at $M_{\rm tot} > M_{\rm gs}$. In Figs. 1(a) and 1(b) with $\delta = 6$ (FIQH limit) at $g = 0.6$ (weak pairing) and $g = 3$ (strong pairing), the lowest-energy states indeed appear at $M_{\rm tot} = 45$ and the expectation values of boson numbers in the lowest-energy states $\langle N_b\rangle$ are close to zero as well. Furthermore, the first excited state is a state with one boson. The energy difference between the ground state and the lowest-energy excited states, mainly due to the chemical potential term, is about 3δ (losing two fermions and gaining one boson), as found in Figs. 1(a) and 1(b).

In the BFQH limit, if the system which has N_b bosons forms a BFQH state with $\nu = \frac{1}{4}$, it will have a Laughlin wave function as

$$\psi_{\nu=\frac{1}{4}}(z_1,z_2,\dots) = \left[\prod_{i<j}^{N_b}(z_i - z_j)^4\right] e^{-\sum_{k=1}^{N}\frac{|z_k|^2}{4l^2}}. \qquad (6)$$

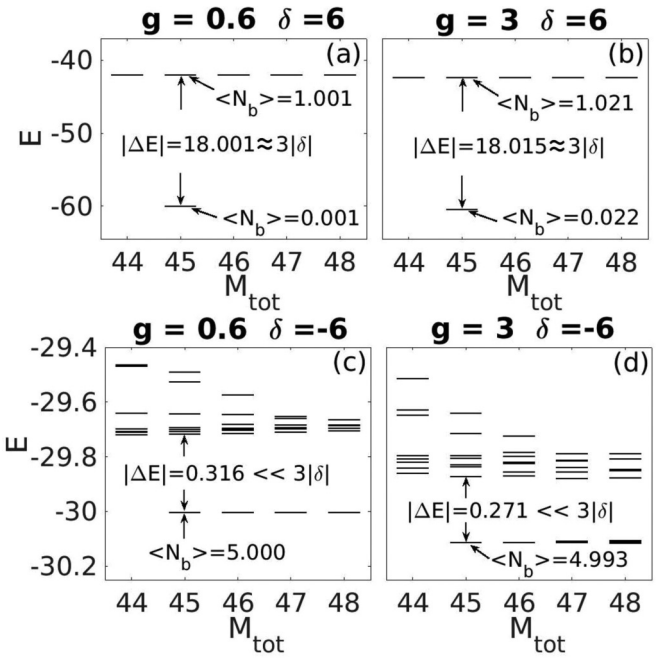

FIG. 1. The energy spectra for a system with $N_{\rm tot} = 10$ fermions, given ten fermionic orbitals and 20 bosonic orbitals. The eight lowest-energy states are plotted for each $M_{\rm tot}$. The ground state at $M_{\rm tot} = M_{\rm gs} = 45$ is separated by a large gap from all excited states for (a) $g = 0.6$ and $\delta = 6$ and (b) $g = 3$ and $\delta = 6$, which are under the FIQH limit. We find a set of low-lying states for $M_{\rm tot} \geqslant M_{\rm gs} = 45$ for (c) $g = 0.6$ and $\delta = -6$ and (d) $g = 3$ and $\delta = -6$, which are under the BFQH limit. These correspond to the Laughlin-like state at $M_{\rm tot} = M_{\rm gs} = 45$ and edge states for $M_{\rm tot} > M_{\rm gs} = 45$. The number of these states indicated in the plots matches the expected number of edge states. $\langle N_b \rangle$'s indicate the expectation values of boson numbers in the corresponding states pointed by small arrows.

The total angular momentum is composed of two parts: the angular momentum of the orbitals occupied by bosons and the internal angular momentum of bosons, $M_{\rm tot} = 4 \times N_b(N_b - 1)/2 + N_b$. In this limit, $N_b = N_{\rm tot}/2$, so $M_{\rm tot} = M_{\rm gs}$, the same as that in the FIQH state. When $M_{\rm tot} < M_{\rm gs}$, there is no state with lower energy than the Laughlin state at $M_{\rm gs}$. However, when $M_{\rm tot} > M_{\rm gs}$, edge states [15] degenerate with the Laughlin state exist. Therefore, by counting and comparing the numbers of low-lying states with the numbers of edge states of Laughlin-type states at various $M_{\rm tot}$, we can demonstrate the system forms a BFQH state. In Figs. 1(c) and 1(d) with $\delta = -6$ at $g = 0.6$ and $g = 3$ we demonstrate this is indeed the case. Notice that the low-lying states are no longer exactly degenerate because of the existence of the pairing interaction. Besides, the boson number in the low-lying state at $M_{\rm tot} = 45$ is very close to 5, the maximum number of bosons, as expected. In Fig. 1(d), we see that $\langle N_b \rangle$ has a bigger deviation from 5. This is due to the fluctuation induced by the strong pairing interaction, namely, fermions have a bigger matrix element to go back and forth between the paired and unpaired states. This is also the reason why the average boson number is bigger in the FIQH state with larger g, as shown in Fig. 1(b). The other thing to notice is in the BFQH regime the low-energy excited states still have essentially all particles as

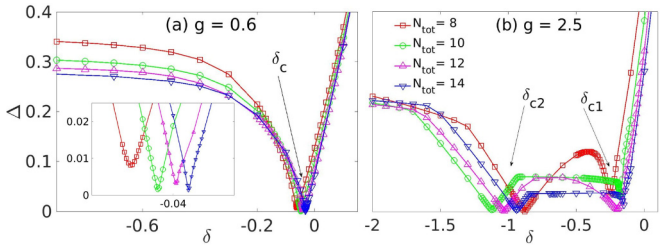

FIG. 2. Plot of the energy gap ($\Delta \equiv E_1 - E_0$, where E_0 and E_1 are the energies of the ground state and the first excited state) vs δ for systems with $N_{\rm tot} = 8$, 10, 12, and 14 at $M_{\rm gs}$ for (a) $g = 0.6$. There is one gap-closing point for each curve. Inset: Blowup of the gap-closing region. (b) For $g = 2.5$. There are two gap-closing points for each curve.

bosons, and as a result their excitation energies in Figs. 1(c) and 1(d) are much smaller than $3|\delta|$ and determined by the boson-boson interaction instead.

In our model, for a specific $N_{\rm tot}$, we need at least $N_{\rm tot}$ fermionic orbitals and $2N_{\rm tot} - 3$ bosonic orbitals, respectively, in order to access the appropriate ground states in the FIQH and BFQH limits. Furthermore, $M_{\rm gs}$ is the same in these two limits. Therefore, we will focus on the $M_{\rm gs}$ sector in our calculations from now on using the minimum orbital numbers mentioned above, unless noted otherwise. To identify the phase boundaries, we drive a system from the FIQH phase to the BFQH phase by changing δ at various g. Since quantum phase transitions between different gapped phases must involve gap closing, we plot the gap Δ of systems in Figs. 2(a) and 2(b) as functions of δ, where the gap is defined as the energy difference between the first excited state and the ground state in the $M_{\rm gs}$ sector. Four system sizes with $N_{\rm tot} = 8$, 10, 12, and 14 are studied. There is one gap-closing point at $g = 0.6$ [weak pairing regime, Fig. 2(a)], indicating a single phase boundary separating the FIQH phase and the BFQH phase. At closer inspection [see the inset of Fig. 2(a)] the gap is not exactly zero in these finite-size calculations, but instead approaches zero with increasing system sizes, indicating the transition is due to a level anticrossing, consistent with a second-order QPT predicted by Barlas and Yang [12]. Unfortunately, the limited system sizes and obvious fluctuations that are present do not allow us to perform finite-size scaling to extract critical exponents.

The situation is quite different at $g = 2.5$ (strong pairing regime). In this case the gap closes twice, implying that the systems undergo two phase transitions in the strong pairing regime. We provide a phase diagram in Fig. 3, based on the locations of the gap-closing points. As predicted by Barlas and Yang [12], there is a direct transition from the FIQH state to the BIQH state in the weak pairing regime (small g). However, it is clear that an unexpected new phase shows up between the FIQH phase and the BFQH phase in the strong pairing regime (large g). We now study the properties of this new phase, and phase transitions involving it.

As discussed earlier, one can distinguish between FIQH and BFQH states by the average number of bosonic molecules in the ground state $\langle N_b \rangle$. To gain intuition into the new phase, we inspect $\langle N_b \rangle$ for both the ground and first excited states.

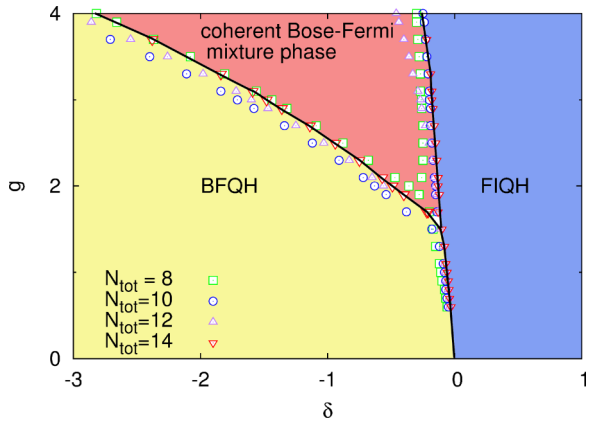

FIG. 3. Phase diagram for systems with $N_{\rm tot} = 8$, 10, 12, and 14. At small g, in the region $\delta > 0$ is the FIQH phase; in the region $\delta < 0$ is the BFQH phase. At large g, the coherent Bose-Fermi mixture phase appears between the FIQH and BFQH phases.

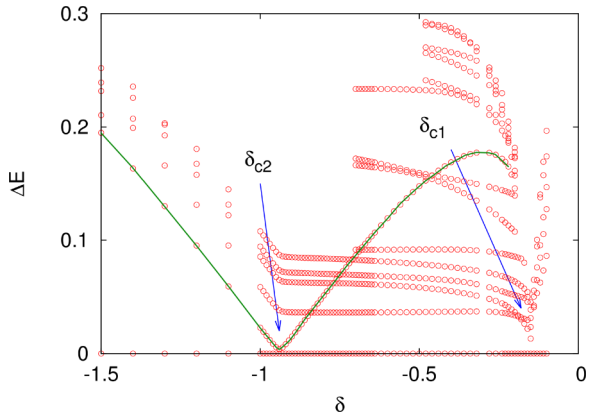

FIG. 5. Energy spectra for the system with $N_{\rm tot} = 14$ at $g = 2.5$. $\Delta E = E_i - E_0$, where E_i denotes the energy of the ith state and E_0 is the ground-state energy.

Figure 4(a) shows that for $N_{\rm tot} = 14$ at $g = 0.6$ (weak pairing regime), $\langle N_b \rangle$'s in the ground state and the first excited state increase monotonically and smoothly from the FIQH state toward the BFQH state, consistent with a second-order phase transition occurring at the critical point δ_c. In the strong pairing regime in Fig. 4(b), $\langle N_b \rangle$ still changes smoothly at δ_{c1}. However, $\langle N_b \rangle$'s in the ground state and the first excited state have jumps at δ_{c2} with opposite signs. This suggests a first-order phase transition triggered by a crossing between the ground and first excited states at δ_{c2}. Motivated by this, we examine the energy spectrum for the $N_{\rm tot} = 14$ case at $g = 2.5$, as shown in Fig. 5. It clearly shows that a high-energy state comes down and eventually becomes the ground state after a sequence of level crossings with other lower-energy states as δ varies from δ_{c1} toward δ_{c2}. This suggests the phase transition occurring at δ_{c2} is a first-order phase transition, triggered by crossing of the Laughlin-like state with the ground state of the new phase between the BFQH state and the FIQH state.

Returning to the boson number expectation value, we expect the ground state in the intermediate phase contains a finite fraction of fermions that stay in a coherent superposition of bound molecular state and unbound scattering state, even when δ is quite negative. Such a superposition state gains energy from the pairing term, and when g is much larger than $v^{(0)}$ and $v^{(2)}$, the pairing term dominates, and the energy gain

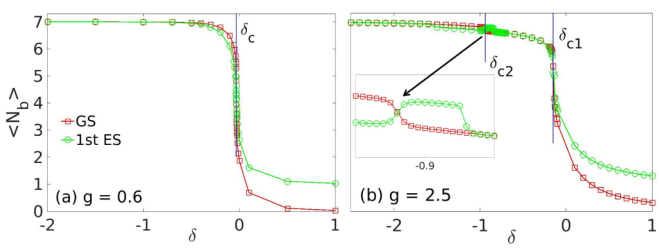

FIG. 4. The expectation values of bosons $\langle N_b \rangle$ in the ground state and the first excited state vs δ for the system with $N_{\rm tot} = 14$ at (a) $g = 0.6$ and (b) $g = 2.5$. The vertical blue lines locate the critical points.

from such coherence dominates the energy cost from putting fermions in the unpaired state. The physics of this intermediate phase is perhaps most clearly revealed by considering the limit $g \to \infty$. In this limit there is no way to tell whether a fermionic atom is in a scattering (open channel, unbound) state, or forms a bound (closed channel, molecular) state with another fermion, because the g term in the Hamiltonian forces all p-wave fermion pairs to be in a coherent superposition of these two states. For this reason we call it the coherent Bose-Fermi mixture phase. A natural variational wave function for the ground state with total angular momentum $M_{\rm tot} = \sum_{\{M\}}(M+1)$ is

$$\psi = \prod_{\{M\}} \left(u_M b_M^\dagger + v_M \sum_{m_1,m_2} \langle m_1, m_2 | 1, M \rangle f_{m_1}^\dagger f_{m_2}^\dagger \right) |0\rangle, \quad (7)$$

where $\{M\}$ is a set of boson angular momenta, with u_M and v_M variational parameters that describe the coherence between the boson and fermion pair states. Since many different configurations of $\{M\}$ exist, the system is most likely compressible. Of course, the actual ground state in the $M_{\rm tot}$ sector can be written as linear superpositions of the state above with different configurations of $\{M\}$ [16].

To test the analysis above, we examine the energy spectrum for $N_{\rm tot} = 14$ of an extreme case with $g = 1$ and all δ and $v^{(0)}$ and $v^{(2)}$ equal to 0 at various $M_{\rm tot}$ in Fig. 6(a). We see the lowest-energy state locates at $M_{\rm gs}$, which is 91 in this case, just as the neighboring BFQH and FIQH phases. The fermion fraction of the total conserved charge is about 0.6 and independent of the system size, as shown in Fig. 6(b). This supports the argument that the g term enforces the ground state to be a superposition of a scattering state and a bound state with $u_M \approx v_M$. On the other hand, the gap $|\Delta E|$ decreases significantly as system size $N_{\rm tot}$ increases; it plausibly extrapolates to zero upon approaching the thermodynamic limit. This strongly suggests that the coherent Bose-Fermi mixture phase is compressible.

We performed a systematic numerical study of the topological phase transition from an integer quantum Hall (FIQH) state made of fermionic atoms, to a bosonic fractional quantum Hall (BFQH) state made of bosonic molecules, driven by

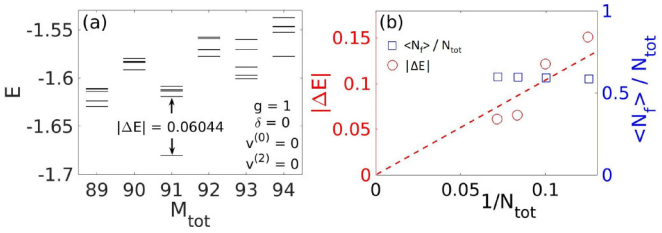

FIG. 6. (a) Energy spectra for $N_{\mathrm{tot}} = 14$ with 14 fermionic orbitals and 25 bosonic orbitals which are the least orbital numbers and parameters $g = 1$, $\delta = 0$, and $v^{(0)} = v^{(2)} = 0$. (b) Energy gap between the lowest-energy state and the first excited state $|\Delta E|$ vs system size on the left axis and average fermion fraction $\frac{\langle N_f \rangle}{N_{\mathrm{tot}}}$ vs system size on the right axis. The black dashed line extrapolates the energy gap to the origin. The fermion fraction is independent of system size.

a p-wave Feshbach resonance. The phase diagram can be separated into two regimes: weak pairing and strong pairing. In the weak pairing regime corresponding to narrow resonance, we demonstrate the existence of a second-order quantum phase transition from the FIQH phase to the BFQH phase, which is consistent with earlier theoretical work [12]. In the strong pairing regime corresponding to wide resonance, a new phase appears which contains a finite fraction of fermions that stay in a coherent superposition of bound molecular states and unbound scattering states. According to the energy spectra and the behavior of $\langle N_b \rangle$, we conclude that the phase transition from the FIQH state to the new phase is of second order; from the new phase to the BFQH phase it is a first-order transition due to the energy level crossing behavior.

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