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Scaling and non-Abelian signature in fractional quantum Hall quasiparticle tunneling amplitude

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New Journal of Physics **13** (2011) 035020 (16pp)

Received 2 December 2010

Published 22 March 2011

Online at <http://www.njp.org/>

doi:10.1088/1367-2630/13/3/035020

Abstract. We study the scaling behavior in the tunneling amplitude when quasiparticles tunnel along a straight path between the two edges of a fractional quantum Hall annulus. Such scaling behavior originates from the propagation and tunneling of charged quasielectrons and quasiholes in an effective field analysis. In the limit when the annulus deforms continuously into a quasi-one-dimensional (1D) ring, we conjecture the exact functional form of the tunneling amplitude for several cases, which reproduces the numerical results in finite systems exactly. The results for Abelian quasiparticle tunneling is consistent with the scaling analysis; this allows for the extraction of the conformal dimensions of the quasiparticles. We analyze the scaling behavior of both Abelian and non-Abelian quasiparticles in the Read–Rezayi \mathbb{Z}_k -parafermion states. Interestingly, the non-Abelian quasiparticle tunneling amplitudes exhibit non-trivial k -dependent corrections to the scaling exponent.

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1. Introduction

Quasiparticle tunneling through narrow constrictions or point contacts that bring counter-propagating edges close could serve as a powerful tool for probing both the bulk topological order and the edge properties of fractional quantum Hall (FQH) liquids (see e.g. [1]). In particular, interference signatures from double point contact devices may reveal the statistical properties of the quasiparticles that tunnel through them [2], especially the non-Abelian ones [3, 4]. In recent interference experiments at the $\nu = 5/2$ FQH state, Willett *et al* [5, 6] found that quasiparticles with charge $e/4$ and $e/2$ both contribute to the interference patterns and dominate in different regimes, which was anticipated in earlier theoretical work [7]. To gain a complete understanding of these experiments, one needs quantitative information about the relative importance of quasiparticles with different charges. Motivated by this, four of the authors and a co-worker [8] performed microscopic calculations of the tunneling matrix elements of various types of quasiparticles, for both the Abelian Laughlin state and the non-Abelian Moore–Read (MR) state. The focus of the previous work was on the dependence of these matrix elements on the tunneling distance: the main result was that the ratio between tunneling matrix elements for quasiparticles with different charges decays with tunneling distance in a Gaussian form, which originates from their charge difference. Such considerations and results are required for a complete understanding of the non-Abelian interferometer [9].

On the other hand, the system size dependence of the tunneling matrix elements is also an interesting issue. In microscopic studies, we start from interacting electrons with fermionic statistics. With proper choices of the microscopic Hamiltonian, ground states with non-trivial topological properties emerge, together with fractionally charged quasiparticle excitations, which may obey either Abelian or non-Abelian statistics. Naturally, in a calculation relevant to quasiparticle tunneling amplitude, we can read out the information about the scaling dimension of the corresponding tunneling operator. In particular, the finite system size cutoff in the numerical calculations may introduce scaling behavior in the tunneling amplitude with an exponent imprinted with the quasiparticle conformal dimension.

In this paper, we study the system size dependence of these matrix elements in the Laughlin and the Moore–Read states. By combining numerical calculations with an effective field theory analysis, we show that their size dependence takes power-law forms with exponents related to the scaling dimensions of the corresponding quasiparticle operators. Furthermore, in the limit when the annulus deforms continuously into a quasi-one-dimensional (1D) ring, we conjecture the precise functional forms of the size dependence, which is not only consistent with the expected power-law form in the scaling limit, but also verified to be true in finite-size systems (using the exact Jack polynomial approach, rather than Lanczos diagonalization with controllable error), indicating their exactness. We also attempt to extend the discussion to the Read–Rezayi states.

We review our model and earlier results in section 2. In section 3, we formulate a scaling theory for the tunneling amplitude of Abelian quasiparticles and compare it with the numerical scaling results. We then conjecture closed-form expressions for the tunneling amplitude, from which we extract exact scaling exponents in section 4. We discuss the scaling behavior for the charge $e/4$ non-Abelian quasihole in the Moore–Read state in section 5 and generalize the discussion to the Read–Rezayi states in section 6. We provide a summary in section 7.

2. Our model and earlier results

In the plane (disc) geometry, we consider an FQH droplet at various filling fractions, which correspond to the series of the Laughlin states, the Moore–Read state and the Read–Rezayi parafermion states. We generate various Abelian and non-Abelian quasiparticles at the center of the droplet. We assume a single-particle tunneling potential

$$V_{\text{tunnel}}(\theta) = V_t \delta(\theta), \quad (1)$$

which breaks the rotational symmetry, where θ is the argument of the complex coordinate $z = x + iy$ in the plane. For the many-body states with N electrons, we write the tunneling operator as the sum of the single-particle operators,

$$\mathcal{T} = \sum_{i=1}^N V_{\text{tunnel}}(\theta_i) = V_t \sum_{i=1}^N \delta(\theta_i). \quad (2)$$

We compute the bulk-to-edge tunneling amplitude $\Gamma^{\text{qh}} = |\langle \Psi_{\text{GS}} | \mathcal{T} | \Psi_{\text{GS}}^{\text{qh}} \rangle|$, where $\Psi_{\text{GS}}^{\text{qh}}$ and Ψ_{GS} are the FQH ground states with and without a quasihole (at the disc center), respectively. As noted in the earlier work [8], the matrix elements consist of contributions from the respective Slater-determinant components $|l_1, \dots, l_N\rangle \in \Psi_{\text{GS}}$ and $|k_1, \dots, k_N\rangle \in \Psi_{\text{GS}}^{\text{qh}}$, where l s and k s are the angular momenta of the occupied orbitals. A non-zero contribution only enters when $|l_1, \dots, l_N\rangle$ and $|k_1, \dots, k_N\rangle$ are identical except for a single pair l_i and k_j with the corresponding angular momentum difference. Therefore, the many-body matrix elements can be decomposed into a sum of single-particle matrix elements

$$v_p(k, l) \equiv \langle k | V_t \delta(\theta) | l \rangle = \frac{V_t}{2\pi} \frac{\Gamma(((k+l)/2) + 1)}{\sqrt{k!l!}}. \quad (3)$$

In the limit that k and l tend to infinity with the tunneling distance fixed at d , i.e. $|k - l| \sim \sqrt{2k}(d/l_B) \ll (k+l)$, the single-particle matrix elements reduce to

$$v_p(k, l) \sim \frac{V_t}{2\pi} e^{-(k-l)^2/4(k+l+2)} \sim \frac{V_t}{2\pi} e^{-d^2/(2l_B)^2}, \quad (4)$$

which reflects the overlap of the two Gaussians separated by a distance d . For more details, see [8]. For convenience, we will henceforth set $V_t = 1$ as the unit of the tunneling amplitudes.

For more relevance to the experimental situations in which quasiparticles tunnel between two edges, we study the edge-to-edge tunneling by inserting n Laughlin quasiholes into the center of the droplet [8]. This transforms a wavefunction $\Psi(\{z_i\})$ to $\prod_{i=1}^N z_i^n \Psi(\{z_i\})$, so that each component Slater determinant becomes a new one, picking up a new normalization factor. The first n orbitals from the center are now completely empty and the electrons are occupying orbitals above n , effectively producing an FQH droplet on an annulus. The tunneling distance $d(n, N)$ between the inner and outer edges decreases monotonically under this transformation. Correspondingly, Γ^{qh} is defined as the edge-to-edge tunneling amplitude. Under the insertion of n Laughlin quasiholes, the single-particle tunneling matrix elements $v_p(k, l)$ become $v_p(n+k, n+l)$. In the limit of $n \gg N$, we have $v_p(n+k, n+l) \rightarrow 1/(2\pi)$.

The earlier work [8] found that the tunneling amplitude ratio of quasiparticles with different charges decays with a Gaussian tail as the inter-edge distance increases. The characteristic length scale associated with this dependence originates partially from the difference in the corresponding quasiparticle charges. In the Moore–Read state, for example, the tunneling amplitude for a charge $e/4$ quasiparticle is larger than that for a charge $e/2$ quasiparticle [8, 9]. Our analyses [8] also show intriguing size dependence in the tunneling amplitudes for the $e/4$ and $e/2$ quasiholes, although their ratio appears to be size independent in the annulus geometry. These observations motivated us to extend the study of the size dependence of Γ^{qh} for different types of quasiholes to the Read–Rezayi series of FQH states, which include the Laughlin and the Moore–Read states as special members.

We note that in equation (2) we introduced the bare tunneling potential for *electrons*, which form FQH liquids. Our results represent the tunneling amplitudes for *quasiparticles* (not for electrons) and have therefore taken into account the many-body correlations of the system. But for quasiparticles, when treated as elementary excitations of the system, these are bare tunneling amplitudes at the microscopic length and energy scales. They are subject to further renormalization when effective low-energy theories are constructed by integrating out degrees of freedom at higher energy and shorter length scales.

3. Field theoretical and numerical analyses of the tunneling amplitudes of Abelian quasiparticles

We start with a field theoretical analysis of the quasiparticle tunneling amplitude, which illustrates our calculation and provides an expectation of the results. We consider, for illustration, a system of electrons and quasiparticles on a cylinder with circumference L and edge-to-edge distance $d \ll L$. This geometry is equivalent to an annulus with an edge-to-edge distance much smaller than the radius. For fixed d , the system size $N \propto L$. We assume that the edge runs around the x -direction, while tunneling occurs along the y -direction at $x = 0$.

We introduce quasiparticle operators $\Psi_{a,j}(x)$, with $j = 1, 2$ corresponding to the two edges, while a is quasiparticle type, and normalize Ψ_a (at each edge) such that the equal time Green's function satisfies

$$G_a(x - x') = \langle 0 | \Psi_a^\dagger(x) \Psi_a(x') | 0 \rangle \sim |x - x'|^{-2\Delta_a}, \quad (5)$$

where Δ_a is the conformal dimension of $\Psi_a(x)$, and proper factors of microscopic length scale ℓ are implied to ensure the correct dimensionality of all quantities.

In a low-energy effective theory, the tunneling Hamiltonian, transferring various types of quasiparticles from one edge to another at $x = 0$, takes the form

$$H_T = L \sum_a t_a [\Psi_{a,1}^\dagger(0) \Psi_{a,2}(0) + \text{h.c.}], \quad (6)$$

where t_a depends on quasiparticle type a but has no L dependence at fixed d . To facilitate comparison with numerical calculations on rotationally invariant geometries, we include a prefactor L —the Jacobian when transforming $\delta(\theta)$ on the annulus to $\delta(x)$ on the cylinder.

A state generated by tunneling a quasiparticle from one edge to another takes the following form (which is a momentum eigenstate),

$$|\Psi_a^{\text{qh}}\rangle = C_a \int_0^L dx dx' \Psi_{a,1}^\dagger(x) \Psi_{a,2}(x') |0\rangle. \quad (7)$$

It is easy to show using equation (5) that the normalization factor $C_a \propto L^{-2+2\Delta_a}$ for

$$\Delta_a \leq 1/2, \quad (8)$$

in which case the corresponding quasiparticle tunneling operator is relevant in the renormalization group (RG) sense [1].

We define the bare quasiparticle tunneling matrix element,

$$\begin{aligned} \Gamma_a &= \langle 0 | H_T | \Psi_a^{\text{qh}} \rangle \\ &\propto t_a L^{-1+2\Delta_a} \int dx dx' \langle 0 | \Psi_{a,2}^\dagger(0) \Psi_{a,1}(0) \Psi_{a,1}^\dagger(x) \Psi_{a,2}(x') | 0 \rangle \\ &= L^{1-2\Delta_a} K_a(d), \end{aligned} \quad (9)$$

where we used properties (5) and (8) and $K_a(d)$ encodes the d dependence of t_a , which is expected to be dominated by the Landau level Gaussian factor [8, 9]. This scaling behavior is expected for ‘elementary’ Abelian quasiholes of the Laughlin type, e.g. the charge $e/3$ quasiholes in the $\nu = 1/3$ Laughlin state, as well as for the charge $e/2$ quasihole (in the identity sector) in the $\nu = 1/2$ Moore–Read state.

We now compare the scaling behavior with numerical results [8]. For clarity, we multiply the tunneling amplitude in figure 4(b) of [8] by a factor of $e^{(d/4l_B)^2}$ (l_B being the magnetic length) for the charge $e/2$ quasihole in the Moore–Read state and plot the rescaled data in figure 1(a). We find that the rescaled data, depending on the corresponding number of electrons N , falls on a series of curves. Assuming that the curves scale as N^α , we obtain $\alpha = 0.47$ for the best scaling collapse, as shown in figure 1(b). Similarly, we analyze and plot the corresponding scaling collapses for charge $e/3$ and $2e/3$ quasiholes in the Laughlin state at $\nu = 1/3$ in figure 2. We obtain the optimal parameter $\alpha = 0.65$ and -0.4 , respectively. In table 1, we compare the optimal fitting α and the conformal dimensions Δ of the corresponding quasiholes. We find excellent to reasonably good agreement with the relation

$$\alpha = 1 - 2\Delta \quad (10)$$

obtained above. In the charge $2e/3$ quasihole case for $\nu = 1/3$, we note that $\Delta = 2/3 > 1/2$ and, therefore, the condition of equation (8) is *not* satisfied. In addition, this is a ‘composite’ (instead of ‘elementary’) quasihole, whose scaling behavior requires a separate (and more complicated) analysis, which we present below.

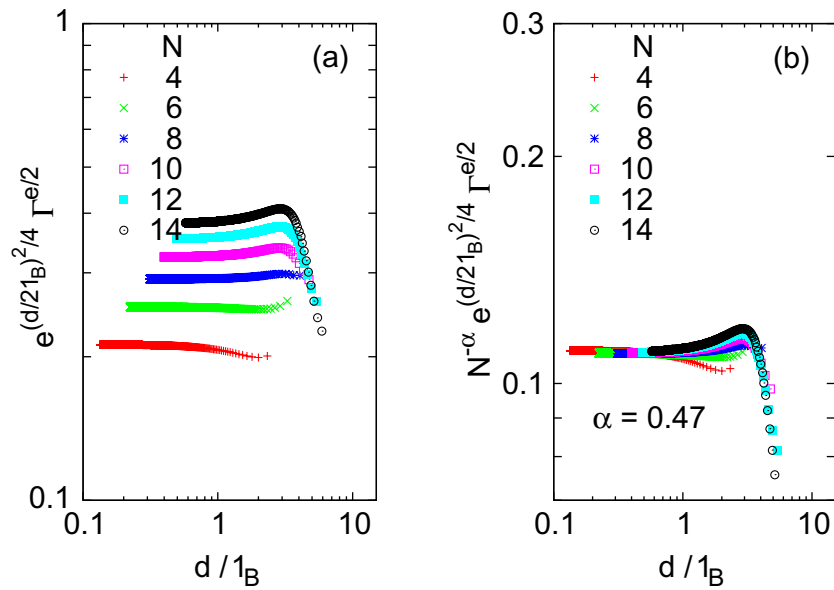


Figure 1. Rescaled tunneling amplitude (a) $e^{(d/4l_B)^2} \Gamma^{e/2}$ and (b) $N^{-\alpha} e^{(d/4l_B)^2} \Gamma^{e/2}$ with $\alpha = 0.47$ for the charge $e/2$ quasihole in the Moore–Read state as a function of the edge-to-edge distance d with $n = 0$ –1000 additional Laughlin $e/2$ quasiholes inserting at the center.

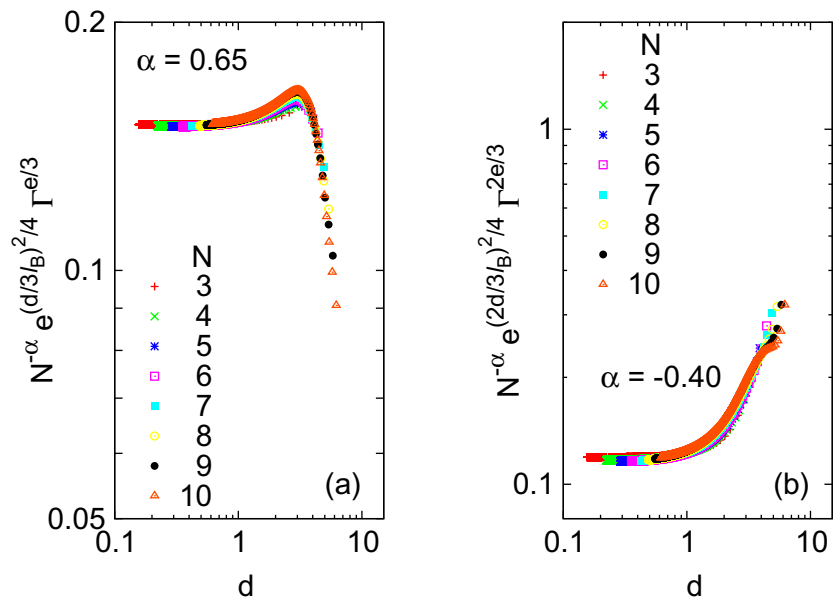


Figure 2. Rescaled tunneling amplitude $N^{-\alpha} e^{(qd/2el_B)^2} \Gamma^q$ for quasiparticles with (a) $q = e/3$, $\alpha = 0.65$ and (b) $q = 2e/3$, $\alpha = -0.4$ in the Laughlin state at $\nu = 1/3$ as a function of the edge-to-edge distance d with $n = 0$ –1000 additional Laughlin $e/3$ quasiholes inserting at the center.

Table 1. The scaling exponent α of the quasihole tunneling amplitude and the corresponding conformal dimension of the quasiholes.

q (ν)	$e/2$ (1/2)	$e/3$ (1/3)	$2e/3$ (1/3)
Δ	1/4	1/6	2/3
$1 - 2\Delta$	1/2	2/3	-1/3
α	0.47	0.65	-0.40

The momentum eigenstate generated by tunneling a $2e/3$ quasihole from one edge to another takes the form

$$|\Psi_a^{2\text{qh}}\rangle = C_{2a} \int dx_1 dx_2 dx'_1 dx'_2 \Psi_{a,1}^\dagger(x_1) \Psi_{a,1}^\dagger(x_2) \Psi_{a,2}(x'_1) \Psi_{a,2}(x'_2) |0\rangle, \quad (11)$$

where Ψ_a is the operator for an $e/3$ quasihole; the expression above explicitly incorporates the fact that the $2e/3$ quasihole is a composite object, and the state created by its tunneling moves two $e/3$ quasiholes from one edge to another, which tunnel simultaneously but are not necessarily bound together before and after the tunneling process.

To calculate the normalization factor C_{2a} and tunneling matrix element $\langle 0|H_T|\Psi_a^{2\text{qh}}\rangle$, we need the full machinery of chiral Luttinger liquid theory for the $\nu = 1/M$ Laughlin state [1], in which $\Psi_a(x) \sim \exp[i\varphi(x)/\sqrt{M}]$ and $\Psi_{2a}(x) \sim \exp[2i\varphi(x)/\sqrt{M}]$, where φ is a bosonic Gaussian field whose normalization is determined by the conformal dimension of Ψ_a , which is $\Delta_a = 1/2M$; we also have $\Delta_{2a} = 4\Delta_a$ following from the fact that φ is a free or Gaussian field. Using the chiral Luttinger liquid theory whose action (for a single edge) takes the form [1]

$$S = \frac{M}{4\pi} \int dt dx [(\partial_t + \nu \partial_x)\varphi(x, t)][\partial_x \varphi(x, t)], \quad (12)$$

it is straightforward to calculate

$$C_{2a} \propto \left| \int dx_1 dx'_1 dx_2 dx'_2 \langle 0|e^{(i/\sqrt{M})[\varphi(x_1)+\varphi(x_2)-\varphi(x'_1)-\varphi(x'_2)]}|0\rangle \right|^{-1} \propto L^{-4+4\Delta_a} \quad (13)$$

and

$$\begin{aligned} \Gamma_{2a} &\equiv \langle 0|H_T|\Psi_a^{2\text{qh}}\rangle \propto t_{2a} L^{-3+4\Delta_a} \left| \int dx dx' \langle 0|e^{(i/\sqrt{M})[2\varphi(0)-\varphi(x)-\varphi(x')]}|0\rangle \right|^2 \\ &= L^{1-8\Delta_a} K_{2a}(d) = L^{1-2\Delta_{2a}} K_{2a}(d), \end{aligned} \quad (14)$$

where we used the fact that $\Delta_{2a} = 4\Delta_a$ in the last step.

Generalizing this analysis to tunneling of a charge me/M quasiparticle in the Laughlin state at $\nu = 1/M$, we find

$$C_{ma} \propto L^{-2m+2m\Delta_a} \quad (15)$$

and

$$\Gamma_{ma} = L^{1-2m^2\Delta_a} K_{ma}(d) = L^{1-2\Delta_{ma}} K_{ma}(d), \quad (16)$$

where we have used the fact that $\Delta_{ma} = m^2\Delta_a$. As a result, relation (10) holds in all of these cases.

4. Conjectures on exact amplitudes in a quasi-one-dimensional (1D) limit

4.1. The quasi-1D limit and the connection with the Jack polynomials

For the Laughlin state and the Moore–Read state, the numerical results presented above agree with the scaling analyses, but not to a high precision. For example, the exponent for the charge $2e/3$ quasihole $\alpha = -0.4$ is 20% smaller than the expectation value of $-1/3$. Clearly, the systems are far from the thermodynamic limit. This motivated us to study the scaling behavior from a different approach: by conjecturing exact (or approximate) formulae and extracting exact exponents from these conjectures. To achieve that, we consider the quasi-1D $d \rightarrow 0$ limit [8], in which the scaling behavior persists, as indicated by figures 1 and 2.

In the mapping from disc to annulus we have described earlier, the wavefunctions, in terms of polynomials of electron coordinates, are unchanged; however, the geometry through the normalization of single-electron basis changes. We point out that in the $d \rightarrow 0$ limit, there is no need to normalize each single-electron Landau level orbital wavefunction by a momentum-dependent coefficient. When both the inner and outer radii are much larger than their difference, the normalization factor depends only on the number of quasiholes in the lowest order, which is the same for all occupied orbitals. From a different point of view, we can write down the antisymmetric many-body ground state and quasihole wavefunctions as weighted sums of Slater determinants $sI_\mu = \det(z_i^{\mu_j})$. In the $d \rightarrow 0$ limit, all Slater determinants are normalizable by the same constant⁸. As a result, the insertion of an additional Abelian quasihole only changes the labels of the orbitals without affecting the amplitude of individual Slater determinants and the overall normalization factor.

With the recent development of the connection [10, 11] of the Jack polynomials (see e.g. [12]) with a negative Jack parameter α_j and FQH wavefunctions, we now understand that these antisymmetric quantum Hall wavefunctions can be written as single Jack polynomials multiplied by the Vandemonde determinant (which are sums of Slater determinants) whose corresponding amplitudes can be evaluated recursively [13]⁹. We emphasize that the amplitudes are integers up to a global normalization constant $1/\sqrt{C}$, where C is an integer. The Jack polynomial connection facilitates *the exact evaluation of the tunneling amplitude* even in relatively large systems. Otherwise, one would need Lanczos diagonalization to produce a numerical approximation with an accuracy that depends on the number of iterations, which is only cost effective for sparse Hamiltonians. For multiparticle interactions, the Hamiltonian becomes very dense and the Lanczos algorithm becomes progressively more expensive. Based on the exact results, we can conjecture [14] the functional forms of the scaling functions for the Laughlin states, the Moore–Read state and the Read–Rezayi \mathbb{Z}_k parafermion states.

4.2. Scaling of quasihole tunneling amplitudes in the Laughlin states

The Laughlin wavefunction at filling fraction $\nu = 1/M$ can be constructed by the chiral boson conformal field (CFT) theory with a compactification radius M [15]. The primary fields are vertex operators $e^{im\varphi(z)/\sqrt{M}}$, where $\varphi(z)$ is the chiral boson. Operators with $m = 1, 2, \dots, M$

⁸ For a concrete example, the four-electron Moore–Read state in the $d \rightarrow 0$ limit, when we set $C = \sqrt{13 \cdot 5!4!3!2!}/\sqrt{12}$ in equation (C4) of [8], contains exactly the same coefficients as the example in [13].

⁹ Generic and unoptimized source codes implementing [13] are available for download from <http://zimp.zju.edu.cn/~xinwan/2010instans/>.

correspond to quasiholes ($m < M$) or electrons ($m = M$), whose conformal dimensions are $\Delta(m, M) = m^2/(2M)$.

For $M = 3$ or $\nu = 1/3$, we conjecture that the tunneling amplitude for the charge $e/3$ quasihole is

$$2\pi\Gamma_{L,M}^{e/M}(N) = \frac{N}{M} B\left(N, \frac{1}{M}\right), \quad (17)$$

where $M = 3$ and N is the number of electrons. Here, we introduce the β function $B(x, \beta) = \Gamma(x)\Gamma(\beta)/\Gamma(x+\beta)$ that for large x and fixed β , asymptotically approaches $\Gamma(\beta)x^{-\beta}$, where $\Gamma(x)$ is the Gamma function (not the tunneling amplitude elsewhere). We verified numerically that the conjecture is *exact for up to ten electrons*; therefore, assuming that the conjecture is exact also for larger systems, we obtain the exact scaling exponent $\alpha^{e/3} = 1 - 1/3 = 2/3$. This is also verified to be correct for $M = 5$.¹⁰ In other words, based on the scaling analysis we discussed earlier, we can compute the conformal dimension of charged Abelian quasiholes in the Laughlin state to be $\Delta_M^1 = 1/(2M)$.

Interestingly, we can make another connection to Jack polynomials by rewriting the tunneling amplitude in a neat way as, e.g. for $\nu = 1/3$,

$$2\pi\Gamma_{L,M=3}^{e/3}(N) = N \frac{\hat{\Omega}(10010010\dots01001)}{\hat{\Omega}(01001001\dots001001)}, \quad (18)$$

where the operator $\hat{\Omega}$ takes the product of the occupied non-zero single-particle momenta, e.g. $\hat{\Omega}(10010010\dots01001) = 3 \cdot 6 \cdot \dots \cdot (3N - 3) = (3N - 3)!!!$. One recognizes that the arguments of $\hat{\Omega}$ are precisely the root configurations of the corresponding Laughlin ground state and the charge $e/3$ quasihole state, which are the final and initial states, respectively, of the quasihole tunneling process. We use the convention throughout the paper to label the leftmost number as the occupation number of the central orbital in the disc geometry.

The exact tunneling amplitude for charge $2e/3$ quasiholes in the Laughlin state discussed earlier can be written as

$$2\pi\Gamma_{L,M=3}^{2e/3}(N) = 2!N \frac{\hat{\Omega}(101101\dots011)}{\hat{\Omega}(011011\dots11011)} = 2!N \frac{\hat{\Omega}\left(\begin{array}{c} 001001\dots01 \\ 100100\dots001 \end{array}\right)}{\hat{\Omega}\left(\begin{array}{c} 010010\dots1001 \\ 001001\dots01001 \end{array}\right)}, \quad (19)$$

where $\hat{\Omega}\left(\begin{smallmatrix} \lambda \\ \mu \end{smallmatrix}\right) = \hat{\Omega}(\lambda)\hat{\Omega}(\mu)$. Apparently, the argument of $\hat{\Omega}$ in the denominator is not the root configuration of the $2e/3$ quasihole state, which should be $001001\dots1001$, nor is that in the numerator the root configuration of the Laughlin ground state. However, if we perform a particle–hole transformation that replaces 1 by 0 in the arguments and vice versa, the argument of $\hat{\Omega}$ in the denominator, *not in the numerator*, becomes the root configuration of the corresponding Laughlin ground state. Meanwhile, the argument of $\hat{\Omega}$ in the numerator becomes the root configuration of the corresponding $e/3$ quasihole state. Therefore, the first equality can be understood as the particle–hole transformation of the charge $e/3$ quasihole tunneling amplitude, implying that the tunneling of a $2e/3$ quasihole is equivalent to the tunneling of an $e/3$ quasiparticle. Then formally, the second equality can be understood as decomposing the $2e/3$ quasihole into two charge $e/3$ quasiholes. By studying $\Gamma_{L,3}^{2e/3}(N+1)/\Gamma_{L,3}^{2e/3}(N)$,

¹⁰ Equation (17) also applies to the integer case ($M = 1$), in which the right-hand side reduces to unity.

we conclude that the scaling behavior of $\Gamma_{L,3}^{2e/3}(N) \sim N^{-1/3}$ is again consistent with equation (10) for

$$\Delta_{L,M}^{me/M} = \frac{m^2}{2M}, \quad (20)$$

as expected. We note that without the exact amplitude conjecture we would obtain a large (20%) error of the exponent based on finite-size scaling only; this means that the systematic error due to finite system size is not negligible in certain cases unless we can conjecture numerically exact results.

We can write down similar results for the $\nu = 1/5$ Laughlin state, which are in agreement with equation (20) with $M = 5$ for $m = 1-4$. For example, for $m = 3$,

$$2\pi\Gamma_{L,5}^{3e/5}(N) = 3!N \frac{\hat{\Omega} \begin{pmatrix} 0001000010\dots001 \\ 0000100001\dots0001 \\ 1000010000\dots00001 \end{pmatrix}}{\hat{\Omega} \begin{pmatrix} 0100001000\dots100001 \\ 0010000100\dots0100001 \\ 0001000010\dots00100001 \end{pmatrix}}. \quad (21)$$

The scaling behavior is asymptotically $\Gamma_{L,5}^{3e/5} \sim N^{-4/5}$, again consistent with equation (10).

4.3. Scaling conjecture for Abelian charge $e/2$ quasiholes in the Moore–Read state

The Moore–Read wavefunction at the filling fraction $\nu = 1/2$ can be constructed by the Ising CFT, which describes the neutral fermion component, and the chiral boson CFT, which describes the charge component [15]. Two quasihole operators relevant to inter-edge tunneling are $\psi_{\text{qh}}^{e/4} = \sigma e^{i\varphi/2\sqrt{2}}$ and $\psi_{\text{qh}}^{e/2} = e^{i\varphi/\sqrt{2}}$. The former is a non-Abelian quasiparticle, whereas the latter an Abelian one. We note that the charge $e/2$ quasihole can be regarded as one of the two fusion results (i.e. $\sigma \times \sigma = 1 + \psi$) of two charge $e/4$ quasiholes; the other, $\psi_{\text{qh}}^{e/2,\psi} = \psi e^{i\varphi/\sqrt{2}}$, is irrelevant (in the renormalization group sense) in inter-edge tunneling. The conformal dimension of the charge $e/2$ quasihole is $\Delta^{e/2} = 1/4$.

We find the tunneling amplitude for $\psi_{\text{qh}}^{e/2}$ in the $d \rightarrow 0$ limit to be *exactly*

$$2\pi\Gamma_{\text{MR}}^{e/2}(N) = N \frac{\hat{\Omega}(11001100110\dots0110011)}{\hat{\Omega}(011001100110\dots0110011)}. \quad (22)$$

This is similar to equation (18) for the $\nu = 1/3$ Laughlin case, emphasizing again the role of root configuration of the states involved in the tunneling process. One can write, equivalently,

$$2\pi\Gamma_{\text{MR}}^{e/2}(N) = \frac{N}{4} B\left(\frac{N}{2}, \frac{1}{2}\right), \quad (23)$$

which leads to $\Gamma_{\text{MR}}^{e/2}(N) \sim N^{1/2}$, again consistent with the scaling analysis, i.e. $\alpha^{e/2} = 1 - 2\Delta^{e/2}$.

5. Scaling analysis for non-Abelian quasiholes in the Moore–Read state

We have seen in the previous two sections that the scaling behavior of the Abelian quasihole tunneling amplitudes can be well understood. The individual scaling exponent is simply related

Table 2. The tunneling amplitude for a charge $e/4$ non-Abelian quasihole in the Moore–Read state. We emphasize that these numbers are exact and can be reproduced identically by equation (24).

N	By Jack polynomial calculation	Evaluation of equation (24)
4	$\frac{8}{\sqrt{33}}$	1.392 621 247 645 583
6	$\frac{32\sqrt{3}}{\sqrt{1067}}$	1.696 791 171 936 646
8	$\frac{512\sqrt{3}}{\sqrt{205\,931}}$	1.954 203 218 066 135
10	$\frac{10\,240}{\sqrt{22\,034\,617}}$	2.181 459 746 630 286
12	$\frac{245\,760}{\sqrt{10\,598\,650\,777}}$	2.387 184 066 751 195
14	$\frac{6\,881\,280}{\sqrt{7\,132\,891\,972\,921}}$	2.576 537 135 673 849
16	$\frac{73\,400\,320\sqrt{3}}{13\sqrt{126\,197\,319\,520\,91}}$	2.752 898 271 693 840
18	$\frac{2\,642\,411\,520\sqrt{3}}{13\sqrt{14\,550\,550\,940\,760\,923}}$	2.918 623 065 050 085

to the conformal dimension of the tunneling particle. In this section, we focus on the non-Abelian charge $e/4$ quasihole in the Moore–Read phase. The quasihole operator can be written as $\Psi_{\text{qh}}^{e/4} = \sigma e^{i\varphi/2\sqrt{2}}$, which consists of a bosonic charge component with conformal dimension $\Delta_c^{e/4} = 1/16$ and a fermionic neutral component also with conformal dimension $\Delta_n^{e/4} = 1/16$. The total dimension is thus $\Delta^{e/4} = \Delta_c^{e/4} + \Delta_n^{e/4} = 1/8$. In some sense, the situation for the charge $e/4$ quasihole in the Moore–Read state is somewhat similar, but not identical to the $2e/3$ quasiparticle at $\nu = 1/3$, as it carries a charge component and a neutral component. It is thus a ‘composite’ object.

Incorporating our prior knowledge of the Abelian cases, we carefully analyze the tunneling amplitude of the non-Abelian quasihole in the quasi-1D limit and conjecture that for the charge $q = e/4$ quasihole in the Moore–Read state with $N = 2n$ electrons, the tunneling amplitude is

$$2\pi\Gamma^{e/4}(N) = \frac{N/2}{4} \sqrt{B\left(\frac{N}{2}, \frac{1}{2} + \frac{\sqrt{3}}{4}\right) B\left(\frac{N}{2}, \frac{1}{2} - \frac{\sqrt{3}}{4}\right)}. \quad (24)$$

The square-root form, which is absent in the Abelian cases, was conceived by noting that the ground state and the state with quasiholes differ by a twist (σ s at the center and along the edge in the latter). Therefore, the two wavefunction normalization constants (square roots of inverse integers) are not equal and the square root does not disappear in the tunneling amplitude. The second arguments of the two β functions turn out to be the roots of $x^2 - x + (1/16) = 0$. We emphasize that the formula is verified to be *exact to machine precision* ($<10^{-15}$) for up to 18 electrons. In table 2, we list the tunneling amplitudes as ratios of integers and square roots of integers next to their numerical values.

The identification of the tunneling amplitudes by equation (24) implies that it has the same scaling behavior $\Gamma_{\text{MR}}^{e/4}(N) \sim N^{1/2}$ as that of the Abelian charge $e/2$ quasiholes. The underlying physics of the result is, however, not as easy to understand as those of the Abelian quasiholes. Apparently, the scaling exponent $\alpha \neq 1 - 2\Delta^{e/4} = 3/4$, as expected from simple dimension counting. We check the reduced tunneling amplitudes at finite edge-to-edge distance d and compare the scaling collapses with $\alpha = 0.5$ and $\alpha = 1 - 2\Delta^{e/4} = 0.75$ in figure 3. We find that the choice of $\alpha = 0.5$ yields a much better scaling collapse, especially for $d < 3l_B$.

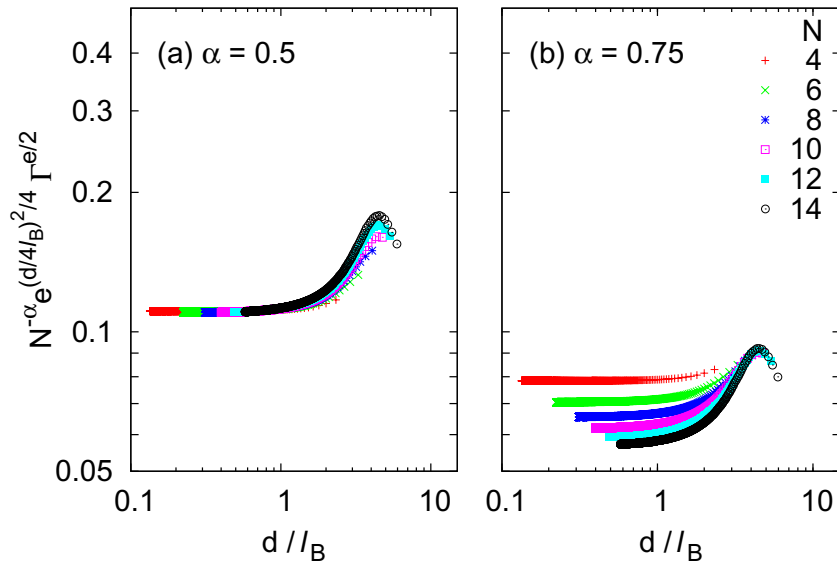


Figure 3. Rescaled tunneling amplitude $N^{-\alpha} e^{(d/4l_B)^2/4} \Gamma^{e/4}$ for charge $e/4$ quasiholes in the Moore–Read state as a function of the edge-to-edge distance d for (a) $\alpha = 0.5$ and (b) $\alpha = 0.75$ with $n = 0$ – 1000 additional Laughlin $e/2$ quasiholes inserted at the center.

While we do not have a satisfactory theory to explain the anomalous scaling behavior for the non-Abelian quasihole, we speculate that one of the potential explanations may be as follows. In the quasi-1D limit, the two edges may not be regarded as independent edges for the neutral component. It is possible that we need to include coupling between neutral components on the two edges (the Abelian charge components are not affected). If the coupling is relevant, we can estimate the length scale for such interactions to be $\sim 3l_B$, which is in agreement with the earlier estimate [16]. Beyond this scale, topological ground state degeneracy and unitary transformation due to braiding are exponentially exact. However, this argument cannot explain why the exponent happens to be $1/2$.

Alternatively, one may speculate that the charge and neutral components may not always be bound together. A realistic tunneling potential, often arising from applying a gate voltage, couples only to the charge component giving neutral components freedom to propagate in the bulk region other than $x = 0$. Qualitatively, we expect that the scaling behavior will be different from simply replacing $\Delta^{e/4}$ with the sum of the charge and neutral conformal dimensions, $\Delta_c^{e/4} + \Delta_n^{e/4}$ in equation (10). In general, the tunneling process may allow additional σ -propagators, which may help to produce the exponent $\alpha = 1/2$ as $\alpha = 1 - 2\Delta_c^{e/4} - 6\Delta_n^{e/4}$ with an anomalous exponent $\delta\alpha = -4\Delta_n^{e/4}$.

6. Speculations on the Read–Rezayi \mathbb{Z}_k parafermion states

To offer additional insights, we attempt to generalize the results to the Read–Rezayi \mathbb{Z}_k parafermion states with the electron operator

$$\psi_e = \psi_1 e^{i\sqrt{((k+2)/k)}\varphi}. \quad (25)$$

Table 3. The tunneling amplitude for charge $ke/(k+2)$ Abelian quasiholes in the Read–Rezayi states. They are all within 1% error of equation (27); the relative errors are listed in parentheses following the tunneling amplitudes.

N/k	$k = 3$	$k = 4$	$k = 5$
2	1.256 203 474 (0.49%)	1.206 153 846 (0.51%)	1.171 688 187 (0.43%)
3	1.451 788 763 (0.65%)	1.358 816 509 (0.65%)	1.296 273 516 (0.52%)
4	1.614 288 884 (0.73%)	1.483 200 501 (0.71%)	1.396 827 446 (0.58%)
5	1.755 379 103 (0.77%)	1.589 612 764 (0.74%)	1.481 715 173 (0.60%)
6	1.881 240 395 (0.79%)	1.683 409 192 (0.75%)	1.555 472 123 (0.58%)
7	1.995 594 026 (0.81%)		

The conformal dimension for ψ_1 is $\frac{k-1}{k}$, whereas for the vertex operator it is $\frac{k+2}{2k}$. The filling fraction is $\nu_k = \frac{k}{k+2}$. In practice, we generate this ground state by a Jack parameter $\alpha_j = -(k+1)$ and the corresponding root configuration of $1^k 001^k 00 \dots 1^k$ (where 1^k means k consecutive 1s) so that there are exactly k 1s in any $(k+2)$ consecutive orbitals.

The charge $\frac{e}{k+2}$ non-Abelian quasihole operator is

$$\psi_{\text{qh}}^{e/(k+2)} = \sigma_1 e^{i\varphi/\sqrt{k(k+2)}}. \quad (26)$$

The conformal dimension for σ_1 is $\Delta_n = \frac{k-1}{2k(k+2)}$ and for the vertex operator it is $\Delta_c = \frac{1}{2k(k+2)}$. One can form an Abelian quasihole of charge $\frac{ke}{k+2}$ by fusing k $\psi_{\text{qh}}^{e/(k+2)}$ quasiholes. The conformal dimension of the Abelian quasihole is $\Delta_{ke/(k+2)} = \frac{k}{2(k+2)}$. The corresponding root configurations for the smallest-charged non-Abelian and Abelian quasiholes are $1^{k-1} 0101^{k-1} 010 \dots 1^{k-1} 01$ and $01^k 001^k 00 \dots 1^k$, respectively. The $e/4$ and $e/2$ quasiholes in the Moore–Read states correspond to the $k = 2$ cases.

From equations (17) and (23), we conjecture that the tunneling amplitude for the charge $\frac{ke}{k+2}$ Abelian quasihole in the filling factor $\nu = \frac{k}{k+2}$ state is

$$2\pi \Gamma_k^{ke/(k+2),1}(N) = \frac{N}{k+2} B\left(\frac{N}{k}, \frac{k}{k+2}\right). \quad (27)$$

Comparing with the numerical results based on recursive construction, we find that equation (27) is *not exact*, but the errors for states ($M = 1$) up to $k = 5$ are *all within 1%* (table 3). This leads to

$$\Gamma_k^{ke/(k+2),1}(N) \sim N^{1-(k/k+2)} \equiv N^{1-2\Delta_{ke/(k+2)}}, \quad (28)$$

which implies $\Delta_c \equiv \Delta_{e/(k+2)} = \frac{1}{2k(k+2)}$. We note that the relative error is *not increasing monotonically*. For $k = 5$, the relative error saturates at $N = 25$ and decreases at $N = 30$. Due to computational limitation, we are unable to verify that this trend is also true for generic k . But the increase in the relative error for $k = 3$ and 4 also slows down notably around $N = 20$.

We want to obtain a similar approximation for the charge $e/(k+2)$ non-Abelian quasihole, so that we can compute the conformal dimension of σ_1 . Ideally, the form should reduce to equation (24) for $k = 2$ and equation (17) for $k = 1$ (i.e. $M = 3$). But with the origin of the numerous parameters in equation (24) unclear, the attempt has not yet been successful. Instead, we fit numerical results to a power law in each case and list the exponents in table 4, in addition

Table 4. The scaling exponent α for the smallest charge $e/(k+2)$ quasihole tunneling amplitude for the Read–Rezayi series. They are obtained from the exact conjectures (for $k = 1–2$) or by data fitting (for $k = 3–5$).

k	1	2	3	4	5
α	2/3	1/2	0.4586	0.4711	0.4792

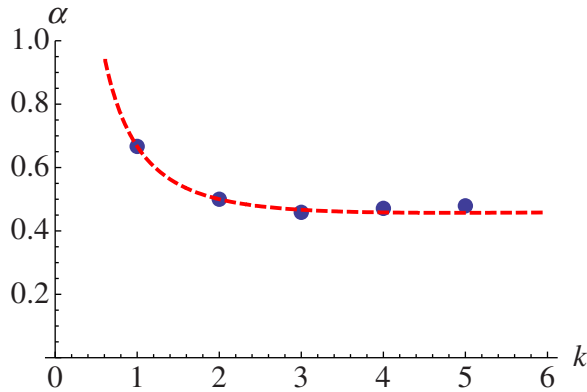


Figure 4. Scaling exponent α for the smallest-charge quasihole tunneling amplitude [$\Gamma(N) \sim N^\alpha$] for the Read–Rezayi series with $k = 1–5$. The dashed line attempts to fit the exponent to a linear dependence on the conformal dimensions of the charge and neutral components (equation (30)).

to the known case of $k = 1$ and 2 for the Read–Rezayi series. In figure 4, we attempt to fit the exponent to the form

$$\alpha^{e/(k+2)} = 1 - (sk + t)\Delta_c - (uk + v)\Delta_n, \quad (29)$$

where s , t , u and v are integers. The linear k dependence in the fitting form takes into account the clustering nature of the Read–Rezayi states (we will return to a discussion of the physics of the hypothesis after we present the fitting result). The result with the best fit is

$$\alpha^{e/(k+2)} = 1 - \frac{k^2 + 3k - 2}{2k(k+2)}, \quad (30)$$

as indicated by the dashed line in figure 4. Interestingly,

$$\alpha^{e/(k+2)} = 1 - 2\Delta_c - 2\Delta_n - \frac{k-1}{2k}. \quad (31)$$

Incidentally, the last term (or the anomalous exponent) is $-(k+2)\Delta_n$. In other words, we find that in equation (29) $s = 0$, $t = 2$, $u = 1$ and $v = 4$.

With the fitting result in hand, let us go back and discuss the physics behind the hypothesis in equation (29) that the scaling exponent has a linear k dependence. First, based on the analysis in the Moore–Read case in section 5, it is natural to expect $\alpha^{e/(k+2)} = 1 - 2\Delta_c - C\Delta_n$, where C is an integer. The reason is that the charge component is constrained by the tunneling potential and we can only have the charge component propagating along the two edges (another $2\Delta_c$ is canceled by the normalization). The fitting result that $s = 0$ and $t = 2$ confirmed this argument.

We cannot say this for the neutral parafermions as there is no such constraint. So we realize that probably we have $C - 2$ more neutral parafermion operators σ_1 related to the propagation along the edge direction than charged vertex operators $e^{i\varphi/\sqrt{k(k+2)}}$ when evaluating the tunneling matrix element. Due to the clustering property of the parafermions, we expect a linear k dependence of C and find $C - 2 = k + 2$. This phenomenologically suggests that the $C - 2$ vertex operators (only tunneling across the edges) together carry one electron charge, which can be justified inside a condensate.

7. Summary and discussion

In summary, we find that the tunneling amplitude for Abelian quasiparticles exhibits finite-size scaling behavior with an exponent related to the conformal dimension of the quasiparticles, irrespective of whether their inter-edge tunneling is relevant or not. This is true for Abelian quasiparticles in both Abelian and non-Abelian quantum Hall states. Generically, we find that in our model, the inter-edge tunneling amplitude for an ideal quasiparticle (arising from the variational wavefunctions) with charge q and a conformal dimension of Δ^q can be expressed as

$$\Gamma^q(N, d) = \Gamma_0 N^{\alpha^q} e^{-(qd/2el_B)^2}, \quad (32)$$

where $\alpha^q = 1 - 2\Delta^q$ for an Abelian quasiparticle with charge component only, e.g. $\alpha^{e/2} = 1/2$ for the charge $e/2$ Abelian quasihole in the Moore–Read state. We note that Γ_0 is related to the propagation of charge bosons and neutral (para)fermions perpendicular to the edges, which contain additional dependence on d as observed for $d > l_B$. The observation of the scaling behavior suggests that the systems are described by underlying conformal field theories; in fact, the conformal dimensions of the Abelian quasiholes obtained from the tunneling amplitudes are in perfect agreement with those in the \mathbb{Z}_k parafermion theories for quantum Hall wavefunctions, based on which we can deduce the conformal dimensions of non-Abelian quasiholes. Computing the conformal dimensions of quasiparticles from wavefunctions has also been attempted in the pattern of zeros classification [17] and in the Jack polynomial approach [18].

The scaling behavior can be alternatively expressed by a differential equation,

$$\frac{\partial \tilde{\Gamma}^q}{\partial l} = \alpha^q \tilde{\Gamma}^q = (1 - 2\Delta^q) \tilde{\Gamma}^q, \quad (33)$$

where $\tilde{\Gamma}^q = e^{(qd/2el_B)^2} \Gamma^q$ and $N = e^l$. Here, we fix the edge-to-edge distance d and the filling fraction ν so the number of electrons $N \sim Ld$, where L is the length of the edge; in the large N limit the annulus is thin, so we do not need to distinguish the lengths of the inner and outer edges. We note that equation (33) resembles the renormalization group flow equation in the context of edge state transport [1]. In particular, $\alpha^{2e/3}$ for the quasiparticles with charge $2e/3$ is negative, which reflects that the quasiparticles are irrelevant to inter-edge tunneling. In the Moore–Read case, on the other hand, quasiparticles with charge $e/4$ and $e/2$ are both relevant, as observed in the interference experiments [5, 6]. It is not clear, however, whether both quasiparticles play a role in the quasiparticle tunneling at a quantum point contact [19].

For the charge $e/4$ non-Abelian quasihole in the Moore–Read state, we find $\alpha^{e/4} = 1/2$ (not $3/4$) and we speculate that the contributions from the charge and neutral components are asymmetric. Interestingly, the scaling exponent coincides with that of the charge $e/2$ Abelian

quasiparticle and therefore we obtain perfect data collapse in figure 5 of [8] for different N . Generically, in the non-Abelian quasiparticle tunneling amplitudes for the Read–Rezayi \mathbb{Z}_k parafermion states, we find anomalous scaling behavior (hence the signature of non-Abelian statistics in model simulations) beyond simple scaling analysis.

Acknowledgments

XW thanks Dimitry Polyakov, Steve Simon, Smitha Vishveshwara and Zhenghan Wang for stimulating discussions. This work was supported by DOE grant no. DE-SC0002140 (ZXH, EHR and KY) and the 973 Program under project no. 2009CB929100 (XW). ZXH, KHL and XW acknowledge the support at the Asia Pacific Center for Theoretical Physics from the Max Planck Society and the Korean Ministry of Education, Science and Technology.

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